MARCH 1999 MONTHLY PROBLEM

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Abstract. [10723] Proposed by *Christopher J. Hillar.* Prove that for all primes $p > 2$,

$$\sum_{i=1}^{p-1} 2^i \cdot i^{p-2} \equiv \sum_{i=1}^{(p-1)/2} i^{p-2} (\bmod p)$$

1. Solution

Proof. First of all, we note the following identity for all n:

$$\sum_{i=1}^{n} \frac{1 + \binom{n}{i} - 2^i}{i} = 0.$$

This can be shown in many ways. One such way is induction. As usual assume it is true for some n, since it's clearly true for $n = 1$. So,

$$\sum_{i=1}^{n+1} \frac{1 + \binom{n+1}{i} - 2^i}{i} = \frac{2 - 2^{n+1}}{n+1} + \sum_{i=1}^{n} \frac{1 + \binom{n}{i} + \binom{n}{i-1} - 2^i}{i}$$

$$= \frac{2 - 2^{n+1}}{n+1} + \sum_{i=1}^{n} \frac{\binom{n}{i-1}}{i} = \frac{2 - 2^{n+1}}{n+1} + \frac{1}{n+1} \sum_{i=1}^{n} \binom{n+1}{i}$$

$$= \frac{2 - 2^{n+1}}{n+1} + \frac{1}{n+1} \left(2^{n+1} - 2\right) = 0.$$

Therefore, the identity is true for all n. So we have that,

$$\sum_{i=1}^{n} 2^i \cdot i^{-1} = \sum_{i=1}^{n} i^{-1} + \sum_{i=1}^{n} \binom{n}{i} \cdot i^{-1}.$$

Now, one can easily see that if $n = p - 1$, where p is prime, then the harmonic sum on the right hand side above is congruent to zero mod p. This is because every integer from 1 to $p - 1$ is covered by some inverse mod p. I.e.

$$\sum_{i=1}^{p-1} i^{-1} \equiv (1 + 2 + 3 + ... + (p-1)) \equiv \frac{p \cdot (p-1)}{2} \equiv 0 (\bmod p).$$

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Since $i^{-1} \equiv i^{p-2} \pmod p$, the original sum is congruent to

$$\sum_{i=1}^{p-1} \left(\begin{array}{c} p-1 \\ i \end{array} \right) \cdot i^{-1} \equiv \sum_{i=1}^{p-1} (-1)^i \cdot i^{-1} \equiv \sum_{i=1}^{p-1} (-1)^i \cdot i^{-1} + \sum_{i=1}^{p-1} i^{-1}$$

$$\equiv 2 \cdot \sum_{i=1}^{(p-1)/2} (2i)^{-1} \equiv \sum_{i=1}^{(p-1)/2} i^{-1} \equiv \sum_{i=1}^{(p-1)/2} i^{p-2} (\mathrm{mod} p).$$

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