MAY 2001 MONTHLY PROBLEM

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ABSTRACT. [10873] Proposed by *B.J. Venkatachala*. Find all non-negative integer solutions to
$$3^m = 2n^2 + 1.$$

1. SOLUTION

All solutions are given by (m, n) = (0,0), (1,1), (2,2), (5,11).

Proof. Suppose m is even. Then, the equation is equivalent to
$$(3^{m/2})^2 - 2n^2 = 1.$$

Setting $t = 3^{m/2}$, this is just the well-known Pell equation. The solutions (t, n) are given by
$$t + n\sqrt{2} = \left(3 + 2\sqrt{2}\right)^k$$

It is a simple matter to see that $(t_k, n_k) = (6t_{k-1} - t_{k-2}, 6n_{k-1} - n_{k-2})$ for $k > 1$, where (t_1, n_1) = (3,2), (t_0, n_0) = (1,0). Hence, mod 9, we have t_k cycles as {1,3,8,0,1,6,8,6,1,0,8,3}. And, mod 11, we have the cycle as {1,3,6,0,5,8,10,8,5,0,6,3}. This shows that if t_k is divisible by 9, it will also be divisible by 11. Therefore, the only solutions with m even are given by (m, n) = (0,0) and (2,2).

Now, suppose m is odd. Hence, we have the Pell equation, where $t = 3^{(m+1)/2}$,
$$t^2 - 6n^2 = 3.$$

Elementary considerations (for instance, see LeVeque, Fundamentals of Number Theory, 1977, p.204), give all solutions to this equation as
$$t + n\sqrt{6} = \left(3 + \sqrt{6}\right)\left(5 + 2\sqrt{6}\right)^k = \left(3 + \sqrt{6}\right)\left(a_n + b_n\sqrt{6}\right)$$

Where $(a_k, b_k) = (10a_{k-1} - a_{k-2}, 10b_{k-1} - b_{k-2})$ for $k > 1$, where (a_1, b_1) = (5,2), (a_0, b_0) = (1,0). Therefore, all solutions (t_k, n_k) are given by $(3a_n + 6b_n, 3b_n + a_n)$. This gives us that $(t_k, n_k) = (10t_{k-1} - t_{k-2}, 10n_{k-1} - n_{k-2})$ for $k > 1$, where (t_1, n_1) = (27,11), (t_0, n_0) = (3,1). An examination as before, this time with mod 7 and mod 9 shows that if t_k $(k > 1)$ is divisible by 9, it will also be divisible by 7. Hence, the only solutions with m odd are (m, n) = (1,1) and (5,11).

$\square$

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