# MATHEMATICAL MONTHLY PROBLEM SOLUTION

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Abstract. [11085]. Proposed by *Matthias Beck, Jesus DeLoera, Mike Develin, and Julian Pfeifle*. Let $d$ be a positive integer, $t_1, \ldots, t_d$ be integers, and $\lambda_1, \ldots, \lambda_d$ be real numbers. Prove that if $\sum_{k=1}^{d} \lambda_k t_k^j$ is an integer for $1 \leq j < d$, then also $\sum_{k=1}^{d} \lambda_k t_k^d$ is an integer.

## 1. Solution

The statement as published is false. Take for example $d = 2$, $t_1 = 2$, $t_2 = -1$, $\lambda_1 = \sqrt{2}/2$, $\lambda_2 = \sqrt{2}$. Then, $\sum_{k=1}^{2} \lambda_k t_k = 0$ is an integer whereas $\sum_{k=1}^{2} \lambda_k t_k^2 = 3\sqrt{2}$ is not. One fix (see below for another) is to add the additional assumption that $\lambda_1 + \cdots + \lambda_d$ is also an integer, and with this in place, we argue as follows.

Let $a_n = \sum_{k=1}^{d} \lambda_k t_k^n$ for $n \geq 1$, and define $a_0 = \lambda_1 + \cdots + \lambda_d$. From the elementary theory of linear recurrences, this sequence satisfies:

$$a_{n+d} = \sum_{i=1}^{d} (-1)^{i-1} e_i a_{n+d-i},$$

in which $e_i = e_i(t_1, \ldots, t_d)$ is the $i$-th elementary symmetric function evaluated at $t_1, \ldots, t_d$. For example, with $d = 2$, this identity simply reads:

$$\lambda_1 t_1^{n+2} + \lambda_2 t_2^{n+2} = (t_1 + t_2)(\lambda_1 t_1^{n+1} + \lambda_2 t_2^{n+1}) - (t_1 t_2)(\lambda_1 t_1^n + \lambda_2 t_2^n).$$

From this identity and the assumptions, it follows easily that $a_n$ is integral for all $n \geq 0$.

*Remark* 1.1. We note that one could also "fix" the statement by stating instead:

Prove that if $\sum_{k=1}^{d} \lambda_k t_k^j$ is an integer for $1 \leq j \leq d$, then also $\sum_{k=1}^{d} \lambda_k t_k^{d+1}$ is an integer.

The recurrence above would also work in this case.

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