AMM11096: THE DETERMINANT AS A POLYNOMIAL IN THE TRACES OF POWERS

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Abstract. [**11096**] Proposed by Said Amghibech. Show that for each positive integer n there exists a polynomial P_n in $\mathbb{C}[x_1, \ldots, x_n]$ such that, for every n-by-n matrix A over $\mathbb{C}$, $\det A = P_n[\mathrm{Tr}A, \mathrm{Tr}A^2, \ldots, \mathrm{Tr}A^n]$.

1. solution

The question may be restated without reference to matrices as follows. Let $R = \mathbb{Q}[\alpha_1, \ldots, \alpha_n]$ be the polynomial ring in algebraically independent indeterminates $\alpha_1, \ldots, \alpha_n$. Let s_m denote the m-th power sum $\sum_{i=1}^{n} \alpha_i^m$. Then, the problem is asking for a polynomial $h(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$ such that

$$\prod_{i=1}^{n} \alpha_i = h(s_1, \ldots, s_n).$$

However, this trivially follows from the (well-known) fact that power sums generate the ring of symmetric functions over $\mathbb{Q}$ (a typical proof uses Newton's identities).

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