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Abstract. [11098]. *Proposed by Christopher Hillar and Darren Rhea.* Let

$$f(n) = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{2^i - 1} \binom{n}{i}.$$

Prove that $f(n) = \Theta(\ln n)$.

1. solution

Proof. Without changing the result, we may prove that $f(n) = \Theta(\log_2 n)$. Examine first a related sum,

$$g(n) = -\sum_{i=1}^{n} \frac{(-2)^i}{2^i - 1} \binom{n}{i}.$$

Notice that

$$\begin{aligned} -g(n) + f(n) &= \sum_{i=1}^{n} (-1)^i \left(\frac{2^i}{2^i - 1} - \frac{1}{2^i - 1} \right) \binom{n}{i} \\ &= \sum_{i=1}^{n} (-1)^i \binom{n}{i} \\ &= (1-1)^n - 1. \end{aligned} \tag{1.1}$$

Thus, $f(n) = g(n) - 1$. It therefore suffices to prove that $g(n) = \Theta(\log_2 n)$. A manipulation of the sum for $g(n)$ gives,

$$\begin{aligned} g(n) &= -\sum_{i=1}^{n} \frac{(-1)^i}{1 - 2^{-i}} \binom{n}{i} \\ &= -\sum_{i=1}^{n} (-1)^i \binom{n}{i} \sum_{j=0}^{\infty} 2^{-ij} \\ &= -\sum_{j=0}^{\infty} \sum_{i=1}^{n} (-2^{-j})^i \binom{n}{i} \\ &= \sum_{j=0}^{\infty} 1 - (1 - 2^{-j})^n. \end{aligned} \tag{1.2}$$

We will first prove the correct upper bound for $g(n)$ using this last expression. Breaking up the sum, we have

$$\sum_{j=0}^{\infty} 1-(1-2^{-j})^n = \sum_{j=0}^{\lfloor \log_2 n \rfloor -1} 1-(1-2^{-j})^n + \sum_{j=\lfloor \log_2 n \rfloor}^{\infty} 1-(1-2^{-j})^n.$$

We estimate the right-most summand. From the monotonicity of a standard limit for $1/e$, we have $\left(1-2^{-j}\right)^{2^j \cdot n/2^j} \geq \left(1-2^{-k}\right)^{2^k \cdot n/2^j}$ for $j \geq k$ and thus (taking $k=1$),

$$\sum_{j=\lfloor \log_2 n \rfloor}^{\infty} 1-(1-2^{-j})^n \leq \sum_{j=\lfloor \log_2 n \rfloor}^{\infty} 1-(1/4)^{n/2^j}. \tag{1.3}$$

Also, since $e^{-x} \geq 1-x$ for $x \geq 0$, it follows that

$$\begin{aligned} \sum_{j=\lfloor \log_2 n \rfloor}^{\infty} 1-(1/4)^{n/2^j} &\leq (\ln 4) \sum_{j=\lfloor \log_2 n \rfloor}^{\infty} n/2^j \\ &\leq (\ln 4) \frac{n}{2^{\lfloor \log_2 n \rfloor -1}} \\ &\leq 4 \ln 4. \end{aligned} \tag{1.4}$$

Next, since each of the $\lfloor \log_2 n \rfloor$ terms in the first sum are bounded by 1, it follows that $g(n) \leq \log_2 n + 4\ln 4$. It remains to prove the correct lower bound. This is somewhat easier, as the following computation illustrates:

$$\begin{aligned} \sum_{j=0}^{\infty} 1-(1-2^{-j})^n &\geq \sum_{j=0}^{\lfloor \log_2 n \rfloor} 1-(1-2^{-j})^n \\ &\geq \sum_{j=0}^{\lfloor \log_2 n \rfloor} 1-e^{-n/2^j} \\ &\geq (\lfloor \log_2 n \rfloor + 1)(1-e^{-n/2^{\lfloor \log_2 n \rfloor}}) \\ &\geq (\log_2 n)(1-e^{-1}). \end{aligned} \tag{1.5}$$

This completes the proof. □

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