AMM11226: EQUATION WITH EXACTLY 3 ZEROES IN [0,1]

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ABSTRACT. [**11226**] Proposed by F. Beaucoup and T. Erdelyi. Let $a_1, \ldots, a_n$ be real numbers, each greater than 1. If $n \geq 2$, show that there is exactly one solution in the interval (0,1) to

$$\prod_{j=1}^{n}(1 - x^{a_j}) = 1 - x.$$

1. SOLUTION

We begin with a few straightforward lemmas.

Lemma 1.1. *If $a > 1$, then for $x \in [0, 1)$, we have*

$$\frac{1-x}{1-x^a} \geq \frac{1}{a}.$$

Proof. Elementary calculus shows that the minimum of $h(x) = (a - ax) - (1 - x^a)$ over $[0, 1]$ is $0 = h(1)$. □

Lemma 1.2. *Let $f(x)$ be a differentiable function on $[0, 1]$ such that $f(0) = f(1) = 0$ and $f'(x_0) < 0$ for every $x_0 \in (0, 1)$ such that $f(x_0) = 0$. Then $f(x)$ has at most one zero in $(0, 1)$.*

Proof. Suppose that there are two zeroes $a < b$ of f in $(0, 1)$; we will derive a contradiction. Let C be the (closed) set of zeroes of f in $[0, 1]$. By the hypotheses, f is not the zero function on $[a, b]$; therefore, let $y \in (a, b)$ be in the (open) complement of C. Consider the set of intervals $[c, y)$ with c a zero of f and choose the one with $c < y$ maximal. Similarly, choose d a zero of f minimal with $y < d$. Since $f'(c) < 0$, we must have that $f(x) < 0$ for all $x \in (c, d)$ (intermediate value theorem). Thus,

$$f'(d) = \lim_{h \to 0} \frac{f(d) - f(d-h)}{h} = \lim_{h \to 0} \frac{-f(d-h)}{h} \geq 0,$$

a contradiction. This proves the lemma. □

Lemma 1.3. *Let $f(x)$ be a differentiable function on $[0, 1]$ such that $f(0) = f(1) = 0$ and $f'(0), f'(1) > 0$. Then $f(x_0) = 0$ for some $x_0 \in (0, 1)$.*

Proof. From the hypotheses, f is positive near 0 and negative near 1. Now apply the intermediate value theorem. □

We may now give a solution to the problem. Consider the differentiable function on $[0, 1]$ given by $g(x) = \prod_{j=1}^{n}(1 - x^{a_j})$, and set $f(x) = g(x) - (1 - x)$. We will first prove that if $x_0 \in (0,1)$ is a solution to $f(x) = 0$, then $f'(x_0) < 0$. It will follow from the lemma above that f has at most 1 zero in $(0,1)$. A computation gives:

$$f'(x) = 1 - \sum_{j=1}^{n} \frac{g(x)}{1-x^{a_j}} a_j x^{a_j - 1}.$$

Suppose that $x_0 \in (0,1)$ is such that $f(x_0) = 0$. Then,

$$\begin{aligned} f'(x_0) &= 1 - \sum_{j=1}^{n} \frac{1-x_0}{1-x_0^{a_j}} (a_j x_0^{a_j-1}) \\ &\leq 1 - \sum_{j=1}^{n} \frac{1}{a_j} (a_j x_0^{a_j-1}) \\ &= 1 - \sum_{j=1}^{n} x_0^{a_j-1}. \end{aligned}$$

From the Weierstrass product inequality, we have that

$$1 - x_0 = g(x_0) = \prod_{j=1}^{n} (1 - x_0^{a_j}) \geq 1 - \sum_{j=1}^{n} x_0^{a_j},$$

equality holding only when $x_0 = 0$ or $x_0 = 1$. Rearranging this expression, it therefore follows that $f'(x_0) < 0$ as desired.

Finally, that there is a zero for f in $(0,1)$ follows from the lemma above since $n \geq 2$ implies that that $f'(0) = f'(1) = 1 > 0$.

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