PUTNAM PROBLEM: POLYNOMIAL PARTIAL DIFFERENTIAL EQUATION

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Abstract. Let R be a commutative ring with characteristic 0 and let P be a polynomial in $R[x_1, \ldots, x_n]$ with the following two properties:

(1) $\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) P = 0,$

(2) $x_1^2 + \cdots + x_n^2$ divides P.

Prove that P is the zero polynomial. [Note: the original problem has $R = \mathbb{R}$].

1. solution

Define ∇^2 to be the operator on the left-hand side of (1) and let $S = x_1^2 + \cdots + x_n^2$. Letting M be any polynomial in $R[x_1, \ldots, x_n]$, we calculate:

$$\begin{aligned} \nabla^2(SM) &= \sum_{i=1}^{n} \frac{\partial^2 SM}{\partial x_i^2} \\ &= S\sum_{i=1}^{n} \frac{\partial^2 M}{\partial x_i^2} + 2Mn + \sum_{i=1}^{n} x_i \frac{\partial M}{\partial x_i}. \end{aligned} \tag{1.1}$$

Suppose that $0 \neq P \in R[x_1, \ldots, x_n]$ has properties (1) and (2) above, and write $P = SQ$ for some polynomial Q. Since (1) is linear, we may sum (1.1) over the monomials M in Q. It should be clear also from (1.1) that the map $M \mapsto \nabla^2(SM)$ sends square-free monomials to square-free monomials and nonsquare-free monomials to nonsquare-free monomials. More specifically, if $I \subseteq \{1, \ldots, n\}$ and $X_I = \prod_{i \in I} x_i$, then

$$\nabla^2(SX_I) = (2n + |I|)X_I.$$

In particular, by property (1), this shows that Q has no square-free monomials.

Order monomials lexicographically with $x_1 < \cdots < x_n$, and let $M = x_1^{r_1} \cdots x_n^{r_n}$ be a monomial occurring in Q. Let d be the largest integer such that $r_d \geq 2$. Equation (1.1) applied to M shows that the smallest monomial occurring in $\nabla^2(SM)$ is $\frac{x_1^2 M}{x_d^2}$. If $M' > M$ is any nonsquare-free monomial with a corresponding d', then

$$\frac{x_1^2 M'}{x_{d'}^2} > \frac{x_1^2 M}{x_d^2}.$$

It follows that the smallest monomial in $\nabla^2(SQ)$ comes from the smallest monomial in Q. This contradicts (1) and completes the proof.

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