

A RESULT ABOUT THE DENSITY OF ITERATED LINE INTERSECTIONS IN THE PLANE

CHRISTOPHER J. HILLAR AND DARREN L. RHEA

ABSTRACT. Let S be a finite set of points in the plane and let $\mathcal{T}(S)$ be the set of intersection points between pairs of lines passing through any two points in S . We characterize all configurations of points S such that iteration of the above operation produces a dense set. We also discuss partial results on the characterization of those finite point-sets with rational coordinates that generate all of \mathbb{Q}^2 through iteration of \mathcal{T} .

1. INTRODUCTION

Let S be a set of points in the plane and let $L = \{L_i\}_{i \in I}$ be the set of lines between pairs of points in S . Consider the following operation on S :

$$(1.1) \quad \mathcal{T}(S) = \bigcup_{i \neq j} L_i \cap L_j \subseteq \mathbb{R}^2.$$

In other words, $\mathcal{T}(S)$ is the set of intersection points between pairs of distinct lines in L . If S consists of n collinear points (or no points at all), then the union above is empty; so to keep the notation consistent, we set $\mathcal{T}(S) = \emptyset$ for these cases.

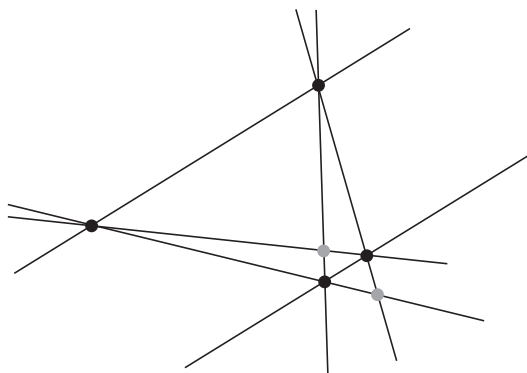


FIGURE 1. $\mathcal{T}(S)$ for a set of points S that form a trapezoid.

As a simple example of the operation \mathcal{T} , let S consist of four black points that are the vertices of a trapezoid as in Figure 1. Then, $\mathcal{T}(S)$ consists of the original

1991 *Mathematics Subject Classification.* 52C30, 52C10.

Key words and phrases. dense planar sets, line intersections, plane geometry.

The work of the first author is supported under a National Science Foundation Graduate Research Fellowship.

four points along with two additional ones shown in gray. It should be clear that for a set of points not all collinear, we have $S \subseteq \mathcal{T}(S)$. Moreover, $\mathcal{T}(S)$ is finite for finite sets S . We are interested here in the iterations, $\mathcal{T}^i(S)$, and specifically, the limiting behavior of such operations on arbitrary finite sets S . The study of such phenomenon naturally leads to the notion of the order of a set S , which we define below. As a matter of convention, we set $\mathcal{T}^0(S) = S$.

Definition 1.1. Let S be a set of points in \mathbb{R}^2 . The *order* of S is the smallest positive integer n such that $\mathcal{T}^n(S) = \mathcal{T}^{n-1}(S)$. If there is no such n , then the order of S is defined to be ∞ .

For example, the order for a set of points forming the vertices of a square is 2. If the order of a set S is 1, then we call S *fixed* under \mathcal{T} . A set S , therefore, has finite order if and only if $\mathcal{T}^n(S)$ is fixed for some nonnegative integer n .

Problem 1.2. *Describe the finite point-sets that have finite order.*

Before discussing the answer to this problem (in Section 2), we describe a non-trivial infinite point-set that has finite order. Let S be the set of rational points on the unit circle, $x^2 + y^2 = 1$. For a given $P \in \mathbb{Q}^2$, choose two points A and B in S such that PA and PB are not tangent to the unit circle. Then, if C and D are the points of intersection of PA and PB (respectively) with the circle, it turns out [7, p. 249] that C and D are both rational. It follows that $P \in \mathcal{T}(S)$ for every $P \in \mathbb{Q}^2$, and thus

$$\mathcal{T}^2(S) = \mathcal{T}(\mathbb{Q}^2) = \mathbb{Q}^2 = \mathcal{T}(S).$$

Excluding the sets of finite order, it follows that iteration of \mathcal{T} produces a strictly increasing chain of sets of points in the plane. In light of this observation, a natural question is whether we arrive at a dense set of points by such a procedure. In other words, is $\bigcup_{i \geq 0} \mathcal{T}^i(S)$ dense in \mathbb{R}^2 ? A more difficult but related question is whether we get all of \mathbb{Q}^2 when S consists of only rational points. We address both of these questions with a complete answer to the first in Section 3 and some partial results for the second in Section 4.

Theorem 1.3. *Let S be a finite set of points in the plane. Then, S has infinite order if and only if $\bigcup_{i \geq 0} \mathcal{T}^i(S)$ is dense in \mathbb{R}^2 .*

The answer to Problem 1.2 found in Corollary 2.3 below, therefore, gives a complete characterization of when iterated line intersections are dense.

Corollary 1.4. *Let S be a finite set of points in the plane. Then, $\bigcup_{i \geq 0} \mathcal{T}^i(S)$ is dense in \mathbb{R}^2 if and only if S is not one of the following sets:*

- (1) *The empty set.*
- (2) *A set of collinear points.*
- (3) *A set of collinear points with one additional noncollinear point.*
- (4) *The vertices of a parallelogram.*
- (5) *The vertices of a parallelogram and the intersection of its two diagonals.*

In the rational case, we conjecture a more exact result.

Conjecture 1.5. *Let S be a finite set of points in the plane with rational coordinates. Then, S has infinite order if and only if $\bigcup_{i \geq 0} \mathcal{T}^i(S) = \mathbb{Q}^2$.*

As a step in the direction of this conjecture, we offer the following; its proof can be found in Section 4.

Theorem 1.6. *Let $R, P, Q, T \in S$ be rational points in the plane with RQ and PT parallel and suppose that RP is not parallel to QT . Then, $\bigcup_{i \geq 0} \mathcal{T}^i(S) = \mathbb{Q}^2$.*

Though we were not motivated by any other particular work, we should remark that a similar question posed by Fejes-Toth (with circles replacing lines) was addressed by Bezdek and Pach in [3], and related results can also be found in the papers [2, 6]. Additionally, Theorem 1.3 has also been discovered recently (independently) by Ismailescu and Radoicic [5].

2. FINITE FIXED SETS

We begin by characterizing sets of finite order. Although one may deduce the main result of this section from Lemmas 3.1 and 3.2 in Section 3, the methods employed here are less cumbersome and might be of independent interest. We will need the following result from elementary geometry.

Theorem 2.1 (The Sylvester-Gallai Theorem). *For every set of n noncollinear points in the plane, there exists a line that contains exactly two of the points.*

Although this fact seems intuitively obvious, its proof eluded even Sylvester, and it was only solved (in published form) some 50 years after being posed by him [4]. We refer the reader to [1] for more details. We are ready to approach Problem 1.2.

Theorem 2.2. *A finite set S fixed under \mathcal{T} must be one of the following configurations:*

- (1) *The empty set.*
- (2) *A set of collinear points with one additional noncollinear point.*
- (3) *The vertices of a parallelogram and the intersection of its two diagonals.*

Proof. Let S be a set of n noncollinear points in the plane that is fixed by \mathcal{T} . Using Theorem 2.1, there exists a line intersecting S in exactly two points P and Q . By assumption, there is some other point X not on this line, and we can choose X so that its altitude from PQ is largest. If all other points lie on the line XP or if all of them lie on XQ , then we are in configuration (2) above. The remaining possibilities break up into two cases.

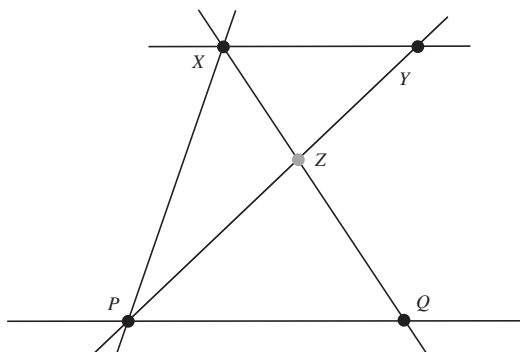


FIGURE 2. Case 1 in the proof of Theorem 2.2

Case 1: There is a point $Y \in S$ not on XP and not on XQ .

We first claim that Y must lie on the line through X that is parallel to PQ . Indeed, any other position for Y would give rise to an intersection between XY and PQ that is not P or Q , contrary to our use of Theorem 2.1 and our assumption that $\mathcal{T}(S) = S$. Relabeling if necessary, Figure 2 depicts the situation. Since S is fixed, the intersection point, Z , of XQ and PY is in S . It follows that XP and YQ must be parallel (otherwise, if W is the intersection point of XP and YQ , then ZW would intersect PQ). Finally, it is easy to see that there can be no other points in S by our choice of P and Q .

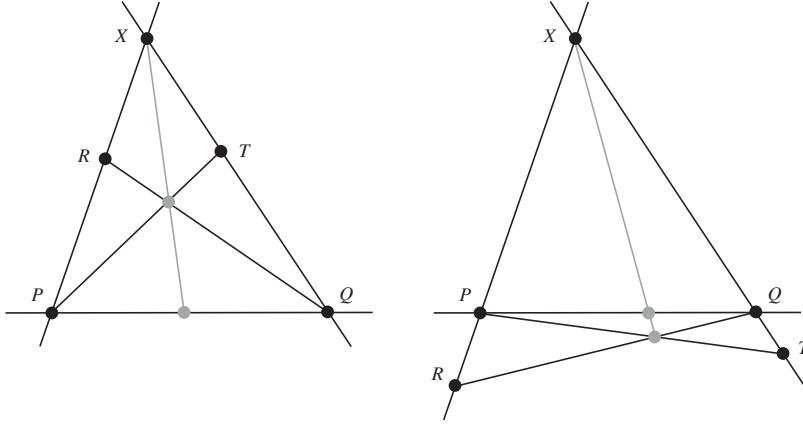


FIGURE 3. Case 2 in the proof of Theorem 2.2

Case 2: Every point in S lies on one of the lines XP or XQ .

If S is not a configuration of type (2), then there are points $R, T \in S$ such that R is on the line XP , T is on the line XQ , and R, T are not X, P , or Q . By the assumption on X and the line PQ , only two configurations for R and T are possible; these are depicted in Figure 3. In both cases, two iterations of \mathcal{T} give rise to a point in S on the line PQ , a contradiction. Therefore, no fixed point-sets other than those of configuration (2) may take this form. This completes the proof. \square

Corollary 2.3. *The finite point-sets with finite order are*

- (1) *The empty set.*
- (2) *A set of collinear points.*
- (3) *A set of collinear points with one additional noncollinear point.*
- (4) *The vertices of a parallelogram.*
- (5) *The vertices of a parallelogram and the intersection of its two diagonals.*

Proof. Let S be a finite set in \mathbb{R}^2 with order n . Applying Theorem 2.2, it follows that $R = \mathcal{T}^{n-1}(S)$ must be one of three types. When R is empty, then S is either itself empty or a set of collinear points. Similarly, a set R of collinear points with one additional point can only be obtained from a set S that is the same as R . Finally, when R forms a parallelogram with the intersection of its diagonals, the set S must either be R or R without its diagonal intersection. \square

3. THE DENSITY THEOREM

Before proving Theorem 1.3, we record the following technical lemmas, the first of which provides a useful characterization of sets of infinite order. For ease of presentation, we say that a point is *strictly contained* in a set K if it is located in its interior.

Lemma 3.1. *Let S be a finite set of infinite order. Then, there exists $n \in \mathbb{N}$ such that $\mathcal{T}^n(S)$ contains a subset of 4 points in which 3 of the points are noncollinear and the fourth point is strictly contained in the triangle determined by these 3 points.*

Proof. We consider the number of vertices v on the convex hull H of S . When $v = 2$, the set S cannot have infinite order. So suppose that $v = 3$. If there is a point of S strictly contained inside H , then we are done. Otherwise, since S has infinite order, there must be two points of S on different edges of H . An iteration of \mathcal{T} then produces our desired point.

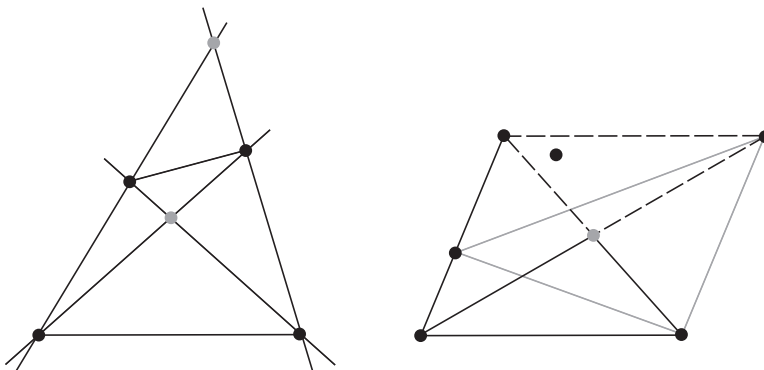


FIGURE 4. Four vertices on the convex hull of S

Assume now that H has exactly four vertices. If these vertices do not form a parallelogram, then one iteration of \mathcal{T} gives us what we want (see Figure 4). Otherwise, there is a point in S which is not a vertex of H and not the intersection of the diagonals of the quadrilateral determined by H . Again in this case, one iteration of \mathcal{T} (giving us the intersection of the two diagonals of H) produces the desired result.

Finally, if $v > 4$, then we proceed as follows. Pick two adjacent vertices A and B . There must be two other vertices C and D such that the edges AB and CD are not parallel (H has at least 5 vertices and is convex). This reduces the problem to the case of 4 vertices not forming a parallelogram (encountered above) and completes the proof of the lemma. \square

Our next result allows one to produce a convergent, nested sequence of triangles.

Lemma 3.2. *Let A , B , and C be noncollinear points, and let P be a point strictly inside $\triangle ABC$. Then, there exist triangles $\triangle A_n B_n C_n$ ($n = 1, 2, \dots$) strictly containing P such that $\lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} B_n = \lim_{n \rightarrow \infty} C_n = P$, and for each n ,*

$$A_n, B_n, C_n \in \bigcup_{j=0}^{\infty} \mathcal{T}^{(j)}(\{A, B, C, P\}).$$

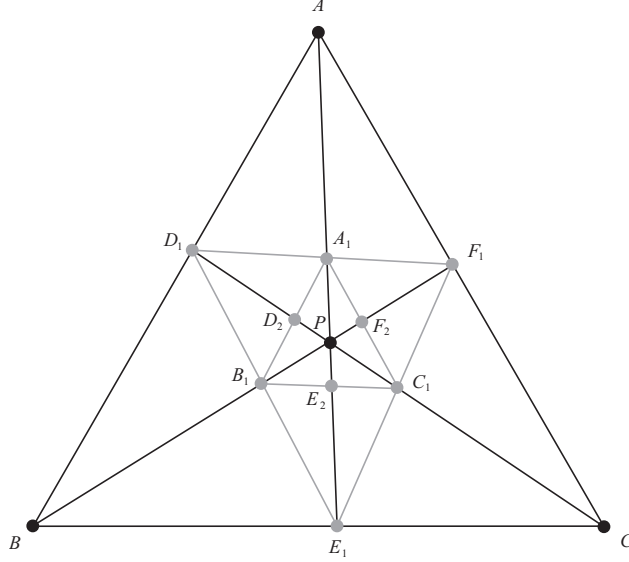


FIGURE 5. Nested triangle iteration

Proof. Given a triangle $\triangle ABC$ and a point P strictly contained in it, we may construct the vertices of another triangle containing this point by intersecting the lines AP , BP , and CP with the edges of $\triangle ABC$. Iterating this procedure produces a nested sequence of triangles strictly containing P with vertices in $\bigcup_{j \geq 0} \mathcal{T}^{(j)}(\{A, B, C, P\})$ (see Figure 5). This sequence contains two types of triangles; we label the odd iterates $\triangle D_n E_n F_n$, while even iterates are denoted by $\triangle A_n B_n C_n$. Here, the A_n (resp. B_n , C_n) are labeled so that they are the ones on the line AP (resp. BP , CP). We claim the vertices of the triangles $\triangle A_n B_n C_n$ all converge to P .

To verify this assertion, it suffices to show that $|A_1 P| < |AA_1|$, $|B_1 P| < |BB_1|$, and $|C_1 P| < |CC_1|$. Without loss of generality, we prove that $|A_1 P| < |AA_1|$. Reducing further, we observe that it is enough to show that the area of $\triangle P D_1 F_1$ is less than the area of $\triangle A D_1 F_1$ (drop altitudes to $D_1 F_1$ from A , P and compare similar triangles). Next, draw the line JK that is parallel to $D_1 F_1$ and passes through P , and label the angles formed as in Figure 6. Since $F_1 P$ and AJ (resp. $D_1 P$ and AK) intersect at B (resp. C), it follows that $\alpha < \beta$ and $\gamma < \delta$. Therefore, when we form the triangle $\triangle Q D_1 F_1$ that is congruent to $\triangle P D_1 F_1$, it must lie entirely inside $\triangle A D_1 F_1$. This finishes the proof. \square

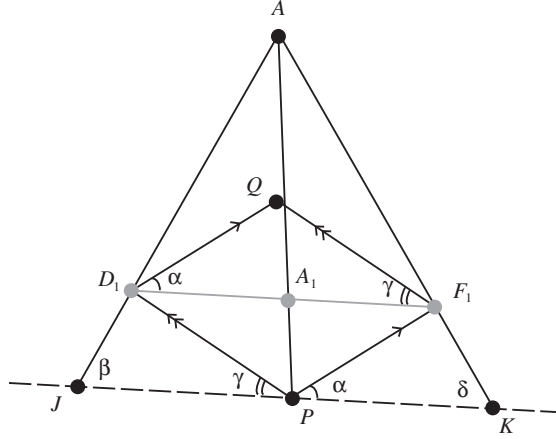


FIGURE 6. Iterations decrease triangle areas

Lemma 3.3. *Let A, B, C be noncollinear points in the plane. If K is a dense set of points in $\triangle ABC$, then $T(K)$ is a dense set of points in the entire plane.*

Proof. Let P be a point in the plane, and let Q_1, Q_2 and R_1, R_2 be points strictly inside $\triangle ABC$ such that Q_1Q_2 and R_1R_2 intersect at P . Since K is dense in $\triangle ABC$, there are a sequence of points $Q_{1n}, Q_{2n} \in K$ and $R_{1n}, R_{2n} \in K$ that converge to Q_1, Q_2 and R_1, R_2 , respectively. Since the intersection of two lines formed by four points is continuous in the four points (the intersection is a rational function in the coordinates of the four points), it follows that the intersections of $Q_{1n}Q_{2n}$ and $R_{1n}R_{2n}$ (which are in $T(K)$) converge to P . This completes the proof. \square

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. The if-direction (\Leftarrow) in the theorem statement is immediate. Therefore, let S be a finite set of infinite order. Using Lemma 3.1, there exists $n \in \mathbb{N}$ such that $T^n(S)$ contains a triangle of vertices and a fourth point strictly contained in the triangle determined by these 3 vertices. We claim that iteration of T on these 4 points produces a dense set of points in the triangle. The theorem then follows from Lemma 3.3.

Let A, B , and C be the vertices of the triangle strictly containing P . Suppose that $K = \bigcup_{j \geq 0} T^j(A, B, C, P)$ does not contain a dense set of points in $\triangle ABC$; we will derive a contradiction. Using Lemma 3.2, we can produce a sequence of triangles, $\triangle A_i B_i C_i$, with vertices in K such these vertices converge to P . Let h be so large that the circle centered at P with radius equal to twice the largest side of $\triangle A_h B_h C_h$ is strictly contained in $\triangle ABC$. Since K is not dense in $\triangle ABC$, it follows that K cannot be dense in $\triangle A_h B_h C_h$ (again using Lemma 3.3).

Let \overline{K} be the closure of K and set $W = \overline{K} \cap \triangle A_h B_h C_h$. Also, let $\text{Int}(\triangle A_h B_h C_h)$ denote the interior of $\triangle A_h B_h C_h$. Since K is not dense in the triangle $\triangle A_h B_h C_h$, the (nonempty) open set $\text{Int}(\triangle A_h B_h C_h) \setminus W$ contains an open ball centered at some point X inside $\triangle A_h B_h C_h$. Consider the set of all closed balls centered at X that do not intersect \overline{K} , and let $r > 0$ denote the supremum over all radii of such balls. The closed ball $\overline{B}(X, r)$ of radius r centered at X must be strictly contained

in $\triangle ABC$ since its interior cannot contain A_h , B_h , or C_h (they are in K) and because of how we chose h .

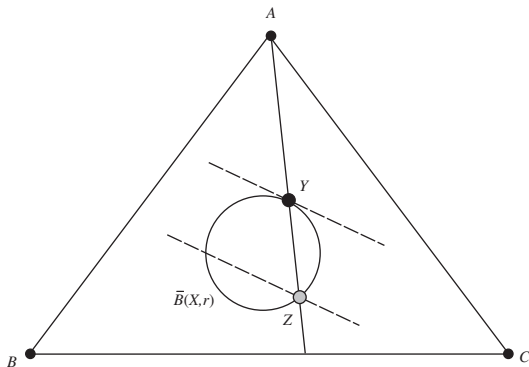


FIGURE 7. Obtaining a contradiction

By construction of $\overline{B}(X, r)$, there exists a point $Y \in \overline{K}$ intersecting the boundary of $\overline{B}(X, r)$. Consider the lines AY , CY , and BY , and notice that they cannot all be tangent to the ball $\overline{B}(X, r)$ (there is only one tangent line through a point on a circle). Therefore, at least one of these lines through Y , say AY , must intersect the interior of $\overline{B}(X, r)$. Let Z be the intersection of the line AY with the boundary of $\overline{B}(X, r)$ (the point Z need not be in \overline{K}). The situation is depicted in Figure 7. The dashed line through Y is the line tangent to the boundary of $\overline{B}(X, r)$ at Y , while the dashed line through Z is parallel to it.

To continue, we observe the following straightforward fact that was discussed in the proof of Lemma 3.3: If $U, V, Q, R \in \overline{K}$ determine two nonparallel lines UV and QR , then the intersection point of UV and QR is in \overline{K} . With this observation in mind, we may use Lemma 3.2 to obtain vertices of triangles $\triangle A'_i B'_i C'_i$ in \overline{K} that contain Y and that also converge to Y . None of the vertices A'_i , B'_i , or C'_i is in the interior of $\overline{B}(X, r)$ by our choice of r .

Finally, we claim that for large enough n , the segment YZ must intersect a side of $\triangle A'_n B'_n C'_n$ in the interior of $\overline{B}(X, r)$, a contradiction to our assumption on r . To see this, notice that for a large n , at least one of the vertices of $\triangle A'_n B'_n C'_n$ must lie between the two parallel lines (depicted in Figure 7) through Y, Z , while none of them will lie beneath the line through Z . It follows that an edge of $\triangle A'_n B'_n C'_n$ intersects the line AY inside $\overline{B}(X, r)$. This contradiction completes the proof. \square

4. THE RATIONAL CASE

We now turn our attention to the case of rational points as in the statement of Conjecture 1.5. We note the following simple observation.

Lemma 4.1. *Suppose that $S = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$ or that $S = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, -1), (1, -2)\}$. Then, $\bigcup_{i \geq 0} \mathcal{T}^i(S) = \mathbb{Q}^2$.*

Proof. Iteration of \mathcal{T} on both sets above gives all of \mathbb{Z}^2 , and it is easily verified that \mathbb{Z}^2 generates all of \mathbb{Q}^2 . \square

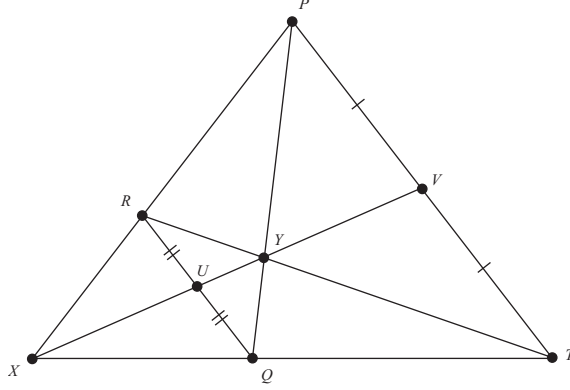


FIGURE 8. Midpoint Lemma 4.2

We next restrict our attention to a particular case involving a pair of parallel lines. We need the following fact from plane geometry.

Lemma 4.2. *Let R, P, Q, T be points in the plane with RQ and PT parallel and suppose that RP is not parallel to QT . Let Y be the intersection of RT and PQ and set X to be the intersection of RP and QT . Then, XY intersects RQ and PT in their midpoints U and V , respectively.*

Proof. Since $\triangle RUY$ and $\triangle TVY$ are similar triangles, we have $RU/TV = UY/VY$. The same reasoning gives us that $UY/VY = UQ/VP$. Examining the large triangles $\triangle XVT$ and $\triangle XPV$, it is also clear that $UQ/TV = XU/XV = RU/VP$. Therefore,

$$UQ = TV \cdot \frac{RU}{VP} = TV^2 \cdot \frac{UQ}{VP^2},$$

so that $TV = VP$. A similar computation shows that $RU = UQ$. \square

We finally arrive at our main result in the rational case. It will be a consequence of Lemma 4.2, and it is the closest we come to proving Conjecture 1.5.

Proof of Theorem 1.6. Since a (rational) translation does not change the problem, we may assume that $Q = (0, 0)$. Moreover, it is easy to see that if $M \in GL_2(\mathbb{Q})$, then

$$M \cdot S = \left\{ M \begin{bmatrix} a \\ b \end{bmatrix} : (a, b) \in S \right\}$$

gives rise to \mathbb{Q}^2 through iteration of \mathcal{T} if and only if S does. Suppose that $R = (a, b)$, $P = (c, d)$, and $T = (u, v)$ with $a, b, c, d, u, v \in \mathbb{Q}$. Since RQ and PT do not define the same line, it follows that $bu - av \neq 0$. Also, since RQ and PT are parallel, we have $bu - av = bc - ad$.

Consider the following matrix:

$$M = \frac{1}{bu - av} \begin{bmatrix} b & -a \\ -v & u \end{bmatrix}.$$

A straightforward computation gives $M \cdot S = \left\{ (0, 0), (0, 1), (1, 0), \left(1, \frac{du - cv}{bu - av}\right) \right\}$. Moreover, since RP is not parallel to QT , it follows that $\frac{du - cv}{bu - av} \neq 1$. Next, set

$\frac{r}{s} = \frac{du-cv}{bu-av}$ in which $r, s \in \mathbb{Z}$ and $\gcd(r, s) = 1$. Suppose first that $r/s > 0$. By successively applying Lemma 4.2, iteration of \mathcal{T} on $M \cdot S$ produces the points:

$$\left\{ \left(0, \frac{l_1}{2^k} \right), \left(1, \frac{rl_2}{s2^k} \right) : l_1, l_2, k \in \mathbb{N}; 0 \leq l_1, l_2 \leq 2^k \right\}.$$

It follows that if we choose k such that $2^{k-1} \geq \max\{r, s\}$, we will have

$$\left\{ (0, 0), \left(0, \frac{r}{2^k} \right), \left(0, \frac{2r}{2^k} \right), (1, 0), \left(1, \frac{r}{2^k} \right), \left(1, \frac{2r}{2^k} \right) \right\} \subseteq \mathcal{T}^k(M \cdot S).$$

Therefore, letting $N = \begin{bmatrix} 1 & 0 \\ 0 & \frac{2^k}{r} \end{bmatrix}$, we must have

$$\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\} \subseteq N \cdot \mathcal{T}^k(M \cdot S).$$

An application of Lemma 4.1 now concludes the proof of this case.

Finally, if $r/s < 0$, then the same examination as above reduces the situation to $S = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, -1), (1, -2)\}$, also covered by Lemma 4.1. \square

5. ACKNOWLEDGMENTS

We would like to thank the anonymous referees for introducing us to the references [2, 3, 5, 6] and for several suggestions that improved the exposition of this paper. Special thanks also go to Kelli Carlson for helping to simplify the proof of Lemma 3.2 and for giving useful comments on preliminary versions of this paper.

REFERENCES

- [1] M. Aigner and G. M. Ziegler, *Proofs from the book*, Springer-Verlag, New York, 1999.
- [2] I. Barany, P. Frankl, H. Maehara, *Reflecting a triangle in the plane*, Graphs and Combinatorics, **9** (2), 97–104, 1993.
- [3] K. Bezdek and J. Pach, *A point set everywhere dense in the plane*, Elem. Math., **40** (4), 81–84, 1985.
- [4] T. Grunwald, *Solution to Problem 4065*, Amer. Math. Monthly, **51**, 169–171, 1944.
- [5] D. Ismailescu and R. Radoicic, *A dense planar point set from iterated line intersections*, Comp. Geom. Theor. Appl., **27** (3), 257–267, 2004.
- [6] J. King, *Three problems in search of a measure*, Amer. Math. Monthly, **101** (7), 609–628, 1994.
- [7] I. Niven, H. Zuckerman, H. Montgomery, *An introduction to the theory of numbers*, John Wiley & Sons, Inc., New York, 1991.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720.
E-mail address: `chillar@math.tamu.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720.
E-mail address: `drhea@math.berkeley.edu`