MINIMAL GENERATORS FOR SYMMETRIC IDEALS

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Abstract. Let $R = K[X]$ be the polynomial ring in infinitely many indeterminates X over a field K, and let $\mathfrak{S}_X$ be the symmetric group of X. The group $\mathfrak{S}_X$ acts naturally on R, and this in turn gives R the structure of a module over the group ring $R[\mathfrak{S}_X]$. A recent theorem of Aschenbrenner and Hillar states that the module R is Noetherian. We address whether submodules of R can have any number of minimal generators, answering this question positively.

Let $R = K[X]$ be the polynomial ring in infinitely many indeterminates X over a field K. Write $\mathfrak{S}_X$ (resp. $\mathfrak{S}_N$) for the symmetric group of X (resp. $\{1, \ldots, N\}$) and $R[\mathfrak{S}_X]$ for its (left) group ring, which acts naturally on R. A *symmetric ideal* $I \subseteq R$ is an $R[\mathfrak{S}_X]$-submodule of R.

Aschenbrenner and Hillar recently proved [1] that all symmetric ideals are finitely generated over $R[\mathfrak{S}_X]$. They were motivated by finiteness questions in chemistry [2] and algebraic statistics [4]. In proving the Noetherianity of R, it was shown that a symmetric ideal I has a special, finite set of generators called a *minimal Gröbner basis*. However, the more basic question whether I is always cyclic (already asked by Josef Schicho [3]) was left unanswered in [1]. Our result addresses a generalization of this important issue.

Theorem 1. *For every positive integer n, there are symmetric ideals of R generated by n polynomials which cannot have fewer than n $R[\mathfrak{S}_X]$-generators.*

In what follows, we work with the set $X = \{x_1, x_2, x_3, \ldots\}$, although as remarked in [1], this is not really a restriction. In this case, $\mathfrak{S}_X$ is naturally identified with $\mathfrak{S}_\infty$, the permutations of the positive integers, and $\sigma x_i = x_{\sigma i}$ for $\sigma \in \mathfrak{S}_\infty$.

Let M be a finite multiset of positive integers and let $i_1, \ldots, i_k$ be the list of its distinct elements, arranged so that $m(i_1) \geq \cdots \geq m(i_k)$, where $m(i_j)$ is the multiplicity of i_j in M. The *type* of M is the vector $\lambda(M) = (m(i_1), m(i_2), \ldots, m(i_k))$. For instance, the multiset $M = \{1, 1, 1, 2, 3, 3\}$ has type $\lambda(M) = (3, 2, 1)$. Multisets are in bijection with monomials of R. Given M, we can construct the monomial:

$$\mathbf{x}_M^{\lambda(M)} = \prod_{j=1}^{k} x_{i_j}^{m(i_j)}.$$

Conversely, given a monomial, the associated multiset is the set of indices appearing in it, along with multiplicities. The action of $\mathfrak{S}_\infty$ on monomials coincides with the natural action of $\mathfrak{S}_\infty$ on multisets M and this action preserves the type of a multiset (resp. monomial). We also note the following elementary fact.

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Lemma 2. *Let $\sigma \in \mathfrak{S}_\infty$ and $f \in R$. Then there exists a positive integer N and $\tau \in \mathfrak{S}_N$ such that $\tau f = \sigma f$.*

Theorem 1 is a direct corollary of the following result.

Theorem 3. *Let $G = \{g_1, \ldots, g_n\}$ be a set of monomials of degree d with distinct types and fix a matrix $C = (c_{ij}) \in K^{n \times n}$ of rank r. Then the submodule $I = \langle f_1, \ldots, f_n \rangle_{R[\mathfrak{S}_\infty]} \subseteq R$ generated by the n polynomials, $f_j = \sum_{i=1}^n c_{ij} g_i$, $(j = 1, \ldots, n)$, cannot have fewer than r $R[\mathfrak{S}_\infty]$-generators.*

Proof. Suppose that $p_1, \ldots, p_k$ are generators for I; we prove that $k \geq r$. Since each $p_l \in I$, it follows that each is a linear combination, over $R[\mathfrak{S}_\infty]$, of monomials in G. Therefore, each monomial occurring in p_l has degree at least d, and, moreover, any degree d monomial in p_l has the same type as one of the monomials in G.

Write each of the monomials in G in the form $g_i = \mathbf{x}_{M_i}^{\lambda_i}$ for multisets $M_1, \ldots, M_n$ with corresponding distinct types $\lambda_1, \ldots, \lambda_n$, and express each generator p_l as:

$$p_l = \sum_{i=1}^{n} \sum_{\lambda(M)=\lambda_i} u_{ilM} \mathbf{x}_M^{\lambda_i} \ + \ q_l, \tag{1}$$

in which $u_{ilM} \in K$ with only finitely many of them nonzero, each monomial in q_l has degree larger than d, and the inner sum is over multisets M with type λ_i.

Since each polynomial in $\{f_1, \ldots, f_n\}$ is a finite linear combination of the p_l, and since only finitely many integers are indices of monomials appearing in $p_1, \ldots, p_k$, we may pick N large enough so that all of these linear combinations can be expressed with coefficients in the subring $R[\mathfrak{S}_N]$ (c.f. Lemma 2). Therefore, we have,

$$f_j = \sum_{l=1}^{k} \sum_{\sigma \in \mathfrak{S}_N} s_{lj\sigma} \sigma p_l, \tag{2}$$

for some polynomials $s_{lj\sigma} \in R$. Substituting (1) into (2) gives us that

$$f_j = \sum_{l=1}^{k} \sum_{\sigma \in \mathfrak{S}_N} \sum_{i=1}^{n} \sum_{\lambda(M)=\lambda_i} v_{lj\sigma} u_{ilM} \mathbf{x}_{\sigma M}^{\lambda_i} \ + h_j,$$

in which each monomial appearing in $h_j \in R$ has degree greater than d and $v_{lj\sigma}$ is the constant term of $s_{lj\sigma}$. Since each f_j has degree d, we have that $h_j = 0$. Thus,

$$\sum_{i=1}^{n} c_{ij} \mathbf{x}_{M_i}^{\lambda_i} = \sum_{l=1}^{k} \sum_{\sigma \in \mathfrak{S}_N} \sum_{i=1}^{n} \sum_{\lambda(M)=\lambda_i} v_{lj\sigma} u_{ilM} \mathbf{x}_{\sigma M}^{\lambda_i}.$$

Next, for a fixed i, take the sum on each side in this last equation of the coefficients of monomials with the type λ_i. This produces the n^2 equations:

$$c_{ij} = \sum_{l=1}^{k} \sum_{\sigma \in \mathfrak{S}_N} \sum_{\lambda(M)=\lambda_i} v_{lj\sigma} u_{ilM} = \sum_{l=1}^{k} \left(\sum_{\lambda(M)=\lambda_i} u_{ilM} \right) \left(\sum_{\sigma \in \mathfrak{S}_N} v_{lj\sigma} \right) = \sum_{l=1}^{k} U_{il} V_{lj},$$

in which $U_{il} = \sum_{\lambda(M)=\lambda_i} u_{ilM}$ and $V_{lj} = \sum_{\sigma \in \mathfrak{S}_N} v_{lj\sigma}$. Set U to be the $n \times k$ matrix (U_{il}) and similarly let V denote the $k \times n$ matrix (V_{lj}). These n^2 equations are represented by the equation $C = UV$, leading to the following chain of inequalities:

$$r = \operatorname{rank}(C) = \operatorname{rank}(UV) \leq \min\{\operatorname{rank}(U), \operatorname{rank}(V)\} \leq \min\{n, k\} \leq k.$$

Therefore, we have $k \geq r$, and this completes the proof. □

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