DISCRIMINANTS OF SYMMETRIC MATRICES

ABSTRACT. The discriminant of a symmetric matrix $A^T = A = [x_{ij}]$ in indeterminates x_{ij} is a sum of squares of polynomials in $\mathbb{Z}[x_{ij} : 1 \leq i \leq j \leq n]$. Our treatment follows that of Roy in [1].

Let $R = \mathbb{Z}[x_{ij} : 1 \leq i \leq j \leq n]$. Given a symmetric matrix $A = [x_{ij}]$ in indeterminates x_{ij}, the discriminant of A is the discriminant of the characteristic polynomial for A. Over an algebraic closure K of the fraction field of R, this may be expressed as

$$\prod_{i<j}(y_i - y_j)^2 = \prod_{i<j}(y_i - y_j) \cdot \prod_{i<j}(y_i - y_j),$$

in which $y_1, \ldots, y_n \in K$ are the eigenvalues of A. Notice that the above expression is also

$$\det(V_n V_n^T) = \det(V_n)^2,$$

where V_n is the vandermond matrix

$$V_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ y_1 & y_2 & \cdots & y_n \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{n-1} & y_2^{n-1} & \cdots & y_n^{n-1} \end{bmatrix}.$$

The matrix $B = V_n V_n^T$ has entries which are the Newton sums of the roots y_i. Of course, the trace of A^k is the kth Newton sum of the y_i. Thus, we have that $B = [\mathrm{tr}(A^{i+j})]$. We will now express B as another product CC^T as follows.

Let E_{ij} be the $n \times n$ matrix with 0s everywhere except for a 1 in the (i,j)th entry. A basis for the symmetric matrices is then given by

$$E_{ii},\ (E_{ij} + E_{ji})/\sqrt{2}\ (i \neq j).$$

So, for example, the matrix

$$A = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix}$$

is represented as

$$x_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_{22} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sqrt{2} \cdot x_{12} \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & \end{bmatrix}.$$

This basis is useful since the inner product of two symmetric matrices P, Q with respect to this basis is simply $\mathrm{tr}[PQ]$, as one can easily check.

For each $i = 0, \ldots, n-1$, express the powers A^i in this basis and place the vectors of coefficients as rows of a matrix C. The entries of the $n \times \binom{n+1}{2}$ matrix C will be in $R[\sqrt{2}]$. For instance, in the $n = 2$ case, this matrix C is

$$C = \begin{bmatrix} 1 & 1 & 0 \\ x_{11} & x_{22} & \sqrt{2} \cdot x_{12} \end{bmatrix}. \tag{0.1}$$

By construction, we have the formal identity,

$$B = [\mathrm{tr}(A^{i+j})] = CC^T.$$

Recall that we are interested in the determinant of B; to compute this in terms of (minors of) the matrix C, we recall the following theorem.

Theorem 0.1 (Cauchy-Binet). *Let C be an $n \times m$ matrix and D be an $m \times n$ matrix. For every $I \subseteq \{1, \ldots, m\}$ of cardinality n, denote by C_I the $n \times n$ matrix obtained by extracting from C the columns with indices in I. Similarly let D^I be the $n \times n$ matrix obtained by extracting from D the rows with indices in I. Then, we have*

$$\det(CD) = \sum_{I \subseteq \{1,\ldots,m\}, |I|=n} \det(C_I)\det(D^I).$$

Restricting to the case of interest, when $D = C^T$, we have that $\det(C_I)\det(D^I) = \det(C_I)^2$. In particular, $\det(B) = \det(CC^T)$ is given by a sum of squares of polynomials in $R[\sqrt{2}]$. To conclude, we make the observation that each $\det(C_I)$ can be expressed as a power of $\sqrt{2}$ multiplied by a polynomial in R; therefore, $\det(C_I)^2$ is a power of 2 times a square in R.

In the $n = 2$ example, the discriminant is the sum of the 2×2 minors of C in (0.1):

$$\begin{aligned} &(x_{22} - x_{11})^2 + (\sqrt{2} \cdot x_{12})^2 + (\sqrt{2} \cdot x_{12})^2 \\ &= (x_{22} - x_{11})^2 + (x_{12})^2 + (x_{12})^2 + (x_{12})^2 + (x_{12})^2, \end{aligned}$$

a sum of squares in $\mathbb{Z}[x_{11}, x_{22}, x_{12}]$.

References

[1] M. Roy, *Subdiscriminants of symmetric matrices are sums of squares*, 2005.